Weaker Adversaries Admit Stronger Guarantees for
Online Covering IPs

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Covering Integer Programs

Covering IP

$$\min \sum_j c_j x_j$$

$\forall i$ $$\sum_j a_{ij} x_j \geq 1$$

$$x_j \in \{0, 1, \ldots\}$$

- unknown instance, RO constraint arrival
- variables can provide only partial coverage

$a_{ij} \in [0,1]$

Idea: change coverage cost measure to include partial coverage, adjust learning rate.

Thm [2]: LoC is $O(\log mn)$ competitive.

Facility Location

Non-Metric Facility Location

facility cost connection cost

$$\min \sum_f D_f x_f + \sum_{c,f} d_{c,f} s_{c,f}$$

$\forall c$ $$\sum_f x_f s_{c,f} \geq 1$$

$$x_f, s_{c,f} \in \{0,1\}$$

open facilities client-facility connections

- clients arrive in RO
- must connect arriving clients to open facilities (and open facilities)

Idea: dynamically adjust the set of facilities which are learned when a client arrives.

Thm: LoC is $O(\log mn)$ competitive.

Random Order Arrivals

Set Cover

$$\min \sum_S c_S x_S$$

$\forall i$ $$\sum_{S \ni i} x_S \geq 1$$

$$x_S \in \{0,1\}$$

- unknown instance
- elements arrive in RO
- must irrevocably cover elements on arrival (by buying sets)
- compare to offline OPT

Thm [2]: LoC is $O(\log mn)$ competitive.

(as compared to $O(\log m \log n)$ in adversarial order [1])

Learn or Cover

LoC (Set Cover)

guess $\beta = c(OPT)$

maintain fractional solution, $x_S^0 \leftarrow \frac{\beta}{c_S m}$

for elements i arriving uncovered:

  sample from x^{t-1} with expected cost κ_i (cost to cover i)

  if i is fractionally uncovered by x^{t-1}:

    perform a MWU:

    $x_S^t \leftarrow x_S^{t-1} \cdot \exp\left(\mathbb{I}\{S \ni i\} \cdot \frac{\kappa_i}{c_S}\right)$

    renormalize: $x^t \leftarrow \frac{\beta}{\|x^t\|_1} x^t$

  cover i at cost κ_i

} Cover

} Learn

weighted KL div. to x^* (fractional OPT) measure of cost to cover remaining $i \in U^t$

Potential function: $$\Phi(x^t) = \sum_S c_S \cdot x_S^* \log\left(\frac{x_S^*}{x_S^t}\right) + \beta \cdot \log\left(\frac{1}{\beta} \sum_{i \in U^t} \kappa_i\right)$$

Idea: if x^t is bad, learning decreases. If good, covering decreases.

Adversarial With a Sample

Set Cover

$$\min \sum_S c_S x_S$$

$\forall i$ $$\sum_{S \ni i} x_S \geq 1$$

$$x_S \in \{0,1\}$$

α $1-\alpha$

- unknown instance, adversarial order
- sample a fraction of elements upfront

$B \sim [n]$, $|B| = \alpha n$

- irrevocably cover remaining elements on (adversarial) arrival

$I = [n] \setminus B$

Thm: LoC is $O\left(\frac{1}{\alpha} \cdot \log mn\right)$ competitive.

Analysis

- run LoC on sampled elements B in RO

$LoC \leftarrow \pi(B)$ for random π (get x_B, $\mathcal{S}(B)$)

- for $i \in I$ do no updates, only cover at cost κ_i

buy $\mathcal{S}(B)$, x_B remains unchanged

- argue that the $i \in I$ arriving uncovered can be charged (in expectation) to B

(at a markup of $1/\alpha$)

References

(1) Noga Alon, Baruch Awerbuch, Yossi Azar, Niv Buchbinder, and Joseph Naor. The online set cover problem. SIAM J. Comput., 39(2):361–370, 2009.

(2) Anupam Gupta, Gregory Kehne, and Roie Levin. Random order online set cover is as easy as offline. In 62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022, pages 1253–1264. IEEE, 2021.